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Common fixed point theorem in complete metric space

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ABSTRACT

The purpose of this article is to prove some common fixed point theorem in complete metric space by using rational type contractive conditions. Our aim of this article is to generalized the results of Jaggi [5] and Jaggi and Das [6].

Keywords: *Fixed point, common fixed point, self mapping and complete metric spaces.*
AMS Subject Classification [2000]: *47H10, 54H25, 46J10, 46J15.*

INTRODUCTION

The well known Banach [1] contraction principle states that "If X is a complete metric space and T is a contraction mapping on X into itself then T has unique fixed point in X". Many researchers worked on this principle. Some of them are as follows;

Kannan [8] proved that "If T is self mapping of a complete metric space X into itself satisfying;

$$d(Tx,Ty) \leq \alpha\, [d(Tx,x) + d(Ty,y)\,] \tag{1}$$

for all $x, y \in X$ and $0 \leq \alpha \leq \frac{1}{2}$ then T has unique fixed point in X.

Fisher B. [3] Proved the result with

$$d(Tx,Ty) \leq [d(x,Ty) + d(Tx,y)\,] \tag{2}$$

for all $x, y \in X$ and $0 \leq \alpha \leq \frac{1}{2}$ then T has unique fixed point in X".

A similar conclusion was also obtained by Chaterjee [2].

In 1977 Jaggi [5] introduced the rational expression first time as ;

$$d(Tx,Ty) \leq \alpha \frac{d(x,Tx)d(y,Ty)}{d(x,y)} + \beta\, d(x,y) \tag{3}$$

for all $x, y \in X$, $x \neq y$ and $0 \leq \alpha + \beta \leq 1$ then T has unique fixed point in X.

In 1980 Jaggi and Das [6] obtained some fixed point theorems with the mapping satisfying:

$$d(Tx,Ty) \leq \alpha \frac{d(x,Tx)d(y,Ty)}{d(x,y)+d(x,Ty)+d(y,Tx)} + \beta d(x,y) \quad (4)$$

for all $x, y \in X$, $x \neq y$, $0 \leq \alpha + \beta \leq 1$ then T has unique fixed point in X.

Definition 1.1: A sequence $\{x_n\}$ in metric space (X, d) is called Cauchy sequence if for each $\varepsilon > 0$ there exists a positive integer n_0 such that

$$m, n \geq n_0 \Rightarrow d(x_m, x_n) < \varepsilon$$

Definition 1.2: A complete metric space is a metric space in which every Cauchy sequence is convergent.

Definition 1.3: A sequence $\{x_n\}$ converges to x if $\lim_{n\to\infty} d(x_n, x) = 0$ in this case x is called a limit of $\{x_n\}$ and we write $x_n \to x$.

RESULTS

In this section we prove some common fixed point theorem for self mapping satisfying rational type contractive mapping. In fact we prove following common fixed point theorems.

Theorem 2.1: Let S, T be any two self mappings of a complete metric space X satisfying the condition

$$\begin{aligned} d(Su,Tv) \leq\ & \alpha_1 \left[\frac{d^2(u,Sw)+d^2(u,v)}{1+d(u,Sw)+d(u,v)}\right] \\ & +\alpha_2 \left[\frac{d^2(v,Tt)+d^2(Sw,Tt)}{1+d(v,Tt)+d(Sw,Tt)}\right] \\ & +\alpha_3\sqrt{d(v,Sw).d(u,Tt)} + \alpha_4[d(u,v)] \end{aligned} \quad (2.1.1)$$

Where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are non negative reals such that $2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 < 1$ then S, T have a unique common fixed point.

Proof: Let x_0 be an arbitrary element of X and we construct a sequence $\{x_n\}$ defined as follows

$Sx_{n-1} = x_n, Tx_n = x_{n+1}, Sx_{n+1} = x_{n+2}, Tx_{n+2} = x_{n+3,\dots\dots}$

And $TSx_{n-1} = x_{n+1}, STx_n = x_{n+2}, TSx_{n+1} = x_{n+3}, STx_{n+2} = x_{n+4,\dots\dots}$

Where $n = 1, 2, 3, \dots$

Now putting $u = Ty$, $v = Sx$, $w = x$ and $t = y$ in (2.1.1) then we have

$$\begin{aligned} d(STy,TSx) \leq\ & \alpha_1 \left[\frac{d^2(Ty,Sx)+d^2(Ty,Sx)}{1+d(Ty,Sx)+d(Ty,Sx)}\right] \\ & +\alpha_2 \left[\frac{d^2(Sx,Ty)+d^2(Sx,Ty)}{1+d(Sx,Ty)+d(Sx,Ty)}\right] \\ & +\alpha_3\sqrt{d(Sx,Sx).d(Ty,Ty)} + \alpha_4[d(Ty,Sx)] \\ d(STy,TSx) \leq\ & 2\alpha_1 d(Sx,Ty) + 2\alpha_2 d(Sx,Ty) \\ & +\alpha_4 d(Sx,Ty) \end{aligned} \quad (2.1.2)$$

Now putting $x = x_{n-1}$ and $y = x_n$ in 2.1.2 then we have

$$\begin{aligned} d(STx_n, TSx_{n-1}) \leq\ & 2\alpha_1 d(Sx_{n-1}, Tx_n) + 2\alpha_2 d(Sx_{n-1}, Tx_n) \\ & +\alpha_4 d(Sx_{n-1}, Tx_n) \end{aligned}$$

$$d(x_{n+2},x_{n+1}) \le 2\alpha_1 d(x_n,x_{n+1}) + 2\alpha_2 d(x_n,x_{n+1}) + \alpha_4 d(x_n,x_{n+1}) \quad (2.1.3)$$

From (2.1.3) we conclude that $d(x_{n-1},x_n)$ decreases with n.

i.e., $d(x_{n-1},x_n) \to d(x_0,x_1)$ as $n \to \infty$

If possible let $d(x_0,x_1) > 0$ and taking limit $n \to \infty$ on (2.1.3) then we have

$$d(x_0,x_1) \le 2\alpha_1 d(x_0,x_1) + 2\alpha_2 d(x_0,x_1) + \alpha_4 d(x_0,x_1) = (2\alpha_1 + 2\alpha_2 + \alpha_4) d(x_0,x_1) < d(x_0,x_1)$$

Since $2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 < 1.$
Which contradiction the fact and hence

$$d(x_0,x_1) = 0\ .$$

Next we shall show that $\{x_n\}$ is Cauchy sequence.

Now,

$$d(x_m,x_n) \le d(x_m,x_{m+1}) + d(x_{m+1},x_{n+1}) + d(x_{n+1},x_n)$$

$$d(x_m,x_n) \le d(x_m,x_{m+1}) + d(x_n,x_{n+1}) + d(Sx_n,Tx_m) \quad (2.1.4)$$

By putting $u = x_n, v = x_m, w = x_{m-1}, t = x_{n-1}$ in (2.1.1) then we have

$$\begin{aligned} d(Sx_n,Tx_m) &\le \alpha_1 \left[\frac{d^2(x_n,Sx_{m-1}) + d^2(x_n,x_m)}{1 + d(x_n,Sx_{m-1}) + d(x_n,x_m)}\right] \\ &+ \alpha_2 \left[\frac{d^2(x_m,Tx_{n-1}) + d^2(Sx_{m-1},Tx_{n-1})}{1 + d(x_m,Tx_{n-1}) + d(Sx_{m-1},Tx_{n-1})}\right] \\ &+ \alpha_3 \sqrt{d(x_m,Sx_{m-1}).\, d(x_n,Tx_{n-1})} \\ &+ \alpha_4 [d(x_n,x_m)] \\ &= \alpha_1 \left[\frac{d^2(x_n,x_m) + d^2(x_n,x_m)}{1 + d(x_n,x_m) + d(x_n,x_m)}\right] \\ &+ \alpha_2 \left[\frac{d(x_m,x_n) + d(x_m,x_n)}{1 + d^2(x_m,x_n) + d^2(x_m,x_n)}\right] \\ &+ \alpha_3 \sqrt{d(x_m,x_m).\, d(x_n,x_n)} + \alpha_4 [d(x_n,x_m)] \\ &= 2\alpha_1 d(x_n,x_m) + 2\alpha_2 d(x_n,x_m) + \alpha_4 d(x_n,x_m) \end{aligned}$$

$$d(Sx_n,Tx_m) \le (2\alpha_1 + 2\alpha_2 + \alpha_4) d(x_n,x_m) \quad (2.1.5)$$

From (2.1.4) and (2.1.5) we have

$$d(x_m,x_n) \le d(x_m,x_{m+1}) + d(x_n,x_{n+1}) + (2\alpha_1 + 2\alpha_2 + \alpha_4) d(x_n,x_m)$$

Letting $m,n \to \infty$ then $d(x_n,x_m) \to 0$
as $2\alpha_1 + 2\alpha_2 + \alpha_4 < 1$

Hence $\{x_n\}$ is a Cauchy sequence.

Now we prove z is a common fixed point of S,T.

By putting $u = z, v = x_{n-1}, w = z$ and $t = x_{n-2}$ in (2.1.1) we have

$$d(Sz,Tx_{n-1}) \leq \alpha_1\left[\frac{d^2(z,Sz)+d^2(z,x_{n-1})}{1+d(z,Sz)+d(z,x_{n-1})}\right]$$
$$+\alpha_2\left[\frac{d^2(x_{n-1},Tx_{n-2})+d^2(Sz,Tx_{n-2})}{1+d(x_{n-1},Tx_{n-2})+d(Sz,Tx_{n-2})}\right]$$
$$+\alpha_3\sqrt{d(x_{n-1},Sz).d(z,Tx_{n-2})}$$
$$+\alpha_4[d(z,x_{n-1})]$$
$$d(Sz,x_n) \leq \alpha_1\left[\frac{d^2(z,Sz)+d^2(z,x_{n-1})}{1+d(z,Sz)+d(z,x_{n-1})}\right]$$
$$+\alpha_2\left[\frac{d^2(x_{n-1},x_{n-1})+d^2(Sz,x_{n-1})}{1+d(x_{n-1},x_{n-1})+d(Sz,x_{n-1})}\right]$$
$$+\alpha_3\sqrt{d(x_{n-1},Sz).d(z,x_{n-1})}+\alpha_4[d(z,x_{n-1})]$$

Letting $n \to \infty$ then we have

$$d(Sz,z) \leq \alpha_1\left[\frac{d^2(z,Sz)+d^2(z,z)}{1+d(z,Sz)+d(z,z)}\right]$$
$$+\alpha_2\left[\frac{d^2(z,z)+d^2(Sz,z)}{1+d(z,z)+d(Sz,z)}\right]$$
$$+\alpha_3\sqrt{d(z,Sz).d(z,z)}+\alpha_4[d(z,z)]$$
$$d(Sz,z) \leq (\alpha_1+\alpha_2)d(Sz,z)$$
$$d(Sz,z) < d(Sz,z)$$

Since $2\alpha_1+2\alpha_2+\alpha_3+\alpha_4<1.$

Which gives $d(Sz,z)=0 \Rightarrow Sz=z$

Thus z is a fixed point of S.

Similarly we can show that z is a fixed point of T.

Hence z is a common fixed point of S,T.

We are taking one another point q which is not equal to z such that $Sq=q=Tq$.

By putting $u=z, v=q, w=q, t=z$ in (2.1.1) then we have

$$d(Sz,Tq) \leq \alpha_1\left[\frac{d^2(z,Sq)+d^2(z,q)}{1+d(z,Sq)+d(z,q)}\right]$$
$$+\alpha_2\left[\frac{d^2(q,Tz)+d^2(Sq,Tz)}{1+d(q,Tz)+d(Sq,Tz)}\right]$$
$$+\alpha_3\sqrt{d(q,Sq).d(z,Tz)}+\alpha_4[d(z,q)]$$
$$d(z,q) \leq \alpha_1\left[\frac{d^2(z,q)+d^2(z,q)}{1+d(z,q)+d(z,q)}\right]$$
$$+\alpha_2\left[\frac{d^2(q,z)+d^2(q,z)}{1+d(q,z)+d(q,z)}\right]$$
$$+\alpha_3\sqrt{d(q,q).d(z,z)}+\alpha_4[d(z,q)]$$
$$d(z,q) \leq (2\alpha_1+2\alpha_2+\alpha_4)d(z,q)$$
$$d(z,q) < d(z,q).$$

Since $2\alpha_1+2\alpha_2+\alpha_3+\alpha_4<1$, which gives $d(z,q)=0 \Rightarrow z=q$. Hence z is unique.
This completes the proof of the theorem.

Corollary 2.2: Let T be self mappings of a complete metric space X satisfying the condition

$$d(Tu,Tv) \le \alpha_1\left[\frac{d^2(u,Tu)+d^2(u,v)}{1+d(u,Tu)+d(u,v)}\right] +\alpha_2\left[\frac{d^2(v,Tu)+d^2(u,Tv)}{1+d(v,Tu)+d(u,Tv)}\right] +\alpha_3\sqrt{d(v,Tu).d(u,Tv)}+\alpha_4[d(u,v)] \quad (2.2.1)$$

For all $u,v,w,t \in X$ where $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ are non negative reals such that $2\alpha_1+2\alpha_2+\alpha_3+\alpha_4<1$ then T has a unique fixed point.

Proof: It is sufficient if we take $S=T$ in Theorem 2.1.

Theorem 3.3: Let S,T,R be any three self mappings of a complete metric space X satisfying the condition

$$d(SRu,TRv) \le \alpha_1\left[\frac{d^2(u,SRw)+d^2(u,TRt)+d^2(u,SRw)}{1+d(u,SRw)+d(u,TRt)+d(u,SRw)}\right] +\alpha_2\left[\frac{d^2(v,SRw)+d^2(u,TRt)+d^2(v,TRt)}{1+d(v,SRw)+d(u,TRt)+d(v,TRt)}\right] +\alpha_3\sqrt{d(v,SRw)d(u,TRt)} +\alpha_4[d(SRw,TRt)]+\alpha_5[d(u,v)] \quad (2.3.1)$$

For $u,v,w,t \in X$, where $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$ are non negative reals such that $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5<1$ then SR,TR have a unique common fixed point.

Proof: Let x_0 be an arbitrary element of X and we construct a sequence $\{x_n\}$ defined as follows

$SRx_{n-1}=x_n, TRx_n=x_{n+1}, SRx_{n+1}=x_{n+2}, TRx_{n+2}=x_{n+3}$

and $TRSRx_{n-1}=x_{n+1}, SRTRx_n=x_{n+2}, TRSRx_{n+1}=x_{n+3}, SRTRx_{n+2}= x_{n+4},$

Where $n=1,2,3,\dots$

Now putting $u=TRy, v=SRx, w=x$ and $t=y$ in (2.3.1) then we have

$$d(SRTRy,TRSRx) \le \alpha_1\left[\frac{d^2(TRy,SRx)+d^2(TRy,TRy)+d^2(TRy,SRx)}{1+d(TRy,SRx)+d(TRy,TRy)+d(TRy,SRx)}\right] +\alpha_2\left[\frac{d^2(SRx,SRx)+d^2(TRy,TRy)+d^2(SRx,TRy)}{1+d(SRx,SRx)+d(TRy,TRy)+d(SRx,TRy)}\right] +\alpha_3\sqrt{d(SRx,SRx).d(TRy,TRy)} +\alpha_4[d(SRx,TRy)]+\alpha_5[d(TRy,SRx)]$$
$$=\alpha_1 d(SRx,TRy)+\alpha_2 d(SRx,TRy) +\alpha_4 d(SRx,TRy)+\alpha_5 d(SRx,TRy)$$

$$\Rightarrow \quad d(SRTRy,TRSRx) \le (\alpha_1+\alpha_2+\alpha_4+\alpha_5)d(SRx,TRy) \quad (2.3.2)$$

Now putting $x=x_{n-1}$ and $y=x_n$ in 2.3.2 then we have

$$d(SRTRx_n,TRSRx_{n-1}) \le (\alpha_1+\alpha_2+\alpha_4+\alpha_5)d(SRx_{n-1},TRx_n)$$

$$\Rightarrow \quad d(x_{n+2},x_{n+1}) \le (\alpha_1+\alpha_2+\alpha_4+\alpha_5)d(x_n,x_{n+1}) \quad (2.3.3)$$

From (2.3.3) we conclude that $d(x_{n-1},x_n)$ decreases with n

i.e. $\quad d(x_{n-1},x_n) \to d(x_0,x_1)$ when $n\to\infty$

If possible let $d(x_0,x_1)>0$ and taking limit $n\to\infty$ on (2.3.3) then we have

$$d(x_0,x_1) \le (\alpha_1+\alpha_2+\alpha_4+\alpha_5)d(x_0,x_1)$$

$$d(x_0, x_1) < d(x_0, x_1)$$

Since $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 < 1$

$$d(x_0, x_1) = 0.$$

Next we shall show that $\{x_n\}$ is Cauchy sequence.

Now $$d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1}) + d(x_{n+1}, x_n)$$

$\Rightarrow$ $$d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_n, x_{n+1}) + d(SRx_n, TRx_m) \qquad (2.3.4)$$

By putting $u = x_n, v = x_m, w = x_{m-1}, t = x_{n-1}$ in (2.3.1) then we have

$$\begin{aligned} d(Sx_n, Tx_m) \leq\; & \alpha_1 \left[\frac{d^2(x_n,SRx_{m-1})+d^2(x_n,TRx_{n-1})+d^2(x_n,SRx_{m-1})}{1+d(x_n,SRx_{m-1})+d(x_n,TRx_{n-1})+d(x_n,SRx_{m-1})}\right] \\ & +\alpha_2 \left[\frac{d^2(x_m,SRx_{m-1})+d^2(x_n,TRx_{n-1})+d^2(x_m,TRx_{n-1})}{1+d(x_m,SRx_{m-1})+d(x_n,TRx_{n-1})+d(x_m,TRx_{n-1})}\right] \\ & +\alpha_3 \sqrt{d(x_m, SRx_{m-1}).\, d(x_n, TRx_{n-1})} \\ & +\alpha_4 [d(SRx_{m-1}, TRx_{n-1})] \\ & = \alpha_1 \left[\frac{d^2(x_n,x_m)+d^2(x_n,x_n)+d^2(x_n,x_m)}{1+d(x_n,x_m)+d(x_n,x_n)+d(x_n,x_m)}\right] \\ & +\alpha_2 \left[\frac{d^2(x_m,x_m)+d^2(x_n,x_n)+d^2(x_m,x_n)}{1+d(x_m,x_m)+d(x_n,x_n)+d(x_m,x_n)}\right] \\ & +\alpha_3 \sqrt{d(x_m, x_m).\, d(x_n, x_n)} \\ & +\alpha_4 [d(x_m, x_n)] + \alpha_5 [d(x_n, x_m)] \\ & = \alpha_1 d(x_n, x_m) + \alpha_2 d(x_n, x_m) + \alpha_4 d(x_m, x_n) \\ & +\alpha_5 d(x_n, x_m) \end{aligned}$$

$$d(Sx_n, Tx_m) \leq (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5) d(x_n, x_m) \qquad (2.3.5)$$

From (2.3.4) and (2.3.5) we have

$$\begin{aligned} d(x_m, x_n) \leq\; & d(x_m, x_{m+1}) + d(x_n, x_{n+1}) \\ & +(\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5) d(x_m, x_n) \end{aligned}$$

Letting $m, n \to \infty$ then $d(x_n, x_m) \to 0$ as $\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 < 1$

Hence $\{x_n\}$ is a Cauchy sequence.

Now we prove z is a common fixed point of SR, TR.

By putting $u = z, v = x_{n-1}, w = z$ and $t = x_{n-2}$ in (2.3.1) we have

$$\begin{aligned} d(SRz, TRx_{n-1}) \leq\; & \alpha_1 \left[\frac{d^2(z,SRz)+d^2(z,TRx_{n-2})+d^2(z,SRz)}{1+d(z,SRz)+d(z,TRx_{n-2})+d(z,SRz)}\right] \\ & +\alpha_2 \left[\frac{d^2(x_{n-1},SRz)+d^2(z,TRx_{n-2})+d^2(x_{n-1},TRx_{n-2})}{1+d(x_{n-1},SRz)+d(z,TRx_{n-2})+d(x_{n-1},TRx_{n-2})}\right] \\ & +\alpha_3 \sqrt{d(x_{n-1}, SRz).\, d(z, TRx_{n-2})} \\ & +\alpha_4 [d(SRz, TRx_{n-2})] + \alpha_5 [d(z, x_{n-1})] \end{aligned}$$

Letting $n \to \infty$ then we have

$$d(SRz, z) \le \alpha_1 \left[\frac{d^2(z,SRz)+d^2(z,z)+d^2(z,SRz)}{1+d(z,SRz)+d(z,z)+d(z,SRz)}\right]$$

$$+\alpha_2 \left[\frac{d^2(z,SRz)+d^2(z,z)+d^2(z,z)}{1+d(z,SRz)+d(z,z)+d(z,z)}\right]$$

$$+\alpha_3 \sqrt{d(z,SRz).\,d(z,z)}$$

$$+\alpha_4[d(SRz,z)] + \alpha_5[d(z,z)]$$

$$d(SRz,z) \le (\alpha_1+\alpha_3+\alpha_4)d(SRz,z)$$

$$d(SRz,z) < d(SRz,z)$$

Since $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 < 1$

Which gives $d(SRz,z) = 0$

$\Rightarrow SRz = z$. Thus z is a fixed point of SR.

Similarly we can show that z is a fixed point of TR. Hence z is a common fixed point of SR, TR.

Now we are taking one another point q which is not equal to z such that $SRq = q = TRq$

By putting $u = z, v = q, w = q, t = z$ in (2.3.1) then we have

$$d(SRz, TRq) \le \alpha_1 \left[\frac{d^2(z,SRq)+d^2(z,TRz)+d^2(z,SRq)}{1+d(z,SRq)+d(z,TRz)+d(z,SRq)}\right]$$
$$+\alpha_2 \left[\frac{d^2(q,Sq)+d^2(q,Tz)+d^2(Sq,Tz)}{1+d(q,Sq)+d(q,Tz)+d(Sq,Tz)}\right]$$
$$+\alpha_3 \sqrt{d(q,SRq).\,d(z,TRz)}$$
$$+\alpha_4[d(SRq,TRz)] + \alpha_5[d(z,q)]$$
$$d(z,q) \le \alpha_1 \left[\frac{d^2(z,q)+d^2(z,z)+d^2(z,q)}{1+d(z,q)+d(z,z)+d(z,q)}\right]$$
$$+\alpha_2 \left[\frac{d^2(q,q)+d^2(z,z)+d^2(q,z)}{1+d(q,q)+d(z,z)+d(q,z)}\right]$$
$$+\alpha_3 \sqrt{d(q,q).\,d(z,z)}$$
$$+\alpha_4[d(q,z)] + \alpha_5[d(z,q)]$$
$$\Rightarrow d(z,q) \le (\alpha_1+\alpha_2+\alpha_4+\alpha_5)d(z,q)$$
$$\Rightarrow d(z,q) < d(z,q)$$

Since $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 < 1$

Which gives $d(z,q) = 0 \Rightarrow z = q$

Hence z is unique.

This completes the proof of theorem.

CONCLUSION

In this present article we prove some common fixed point theorem satisfying new rational contractive conditions in metric spaces. In fact our main result is more general then other previous known results.

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