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Determining of temperature field in a L-shaped domain

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ABSTRACT

This paper deals with solving Dirichlet's problem with nonhomogenous boundary conditions, using the method of large singular finite elements for Laplace's equation in a L-shaped domain. This method is particularly suitable for solving singular problems since the analytical form of singularities has been integrated in the approximate solution. It is unique in that the equations are exactly verified except on those areas where internal segments are approached with great precision. Results are compared with those obtained through finite elements method by using the COMSOL software. They dealt with solution u values and those of its first derivatives. Both methods give results that align quite well everywhere, except near singularities where significant differences exist.

Keywords: Singular points, least squares, large elements, finite elements.

INTRODUCTION

The general theory of solutions to Laplace's equation is known as potential theory. The solutions of Laplace's equation are the harmonic functions, which are important in many fields of science, notably the fields of electromagnetism [1] and fluid dynamics [2], because they can be used to accurately describe the behavior of electric, gravitational, and fluid potentials. In the study of heat conduction [3, 4], the Laplace equation is the steady-state heat equation. Numerical solving of Laplace's equation still remains very difficult and usual methods of finite elements and finite differences give unsatisfactory results when they are used in their standard form. These methods as evidenced by various authors [5-12] can be significantly improved if they take into consideration the analytical form of the solution near of the singularities. We therefore apply the method of the large singular finite elements [13] to calculate the temperature field in polygon L-shaped with Dirichlet non-homogenous boundary conditions. This gives good results all over the study domain while the finite element method gives good results only on areas located far from singularities. This shows the power, efficiency and accuracy of the method of large singular finite elements for a limited number of conserved coefficients than the finite elements method.

MATERIALS AND METHODS

Either to determine the stationary temperature field u in a L-shaped polygon comprising eight segments of unit length kept at various constant temperatures. In reduced variables, the problem consists in solving Laplace's equation with Dirichlet boundary conditions with jumps. The L-shaped domain [14] of side 2 and boundary conditions are given in figure 1. The problem as posed is singular due to discontinuities in limit conditions on borders and the six summits of the domain where there is discontinuity in the external normal in the border of the domain. This is solved using method of large singular finite elements.

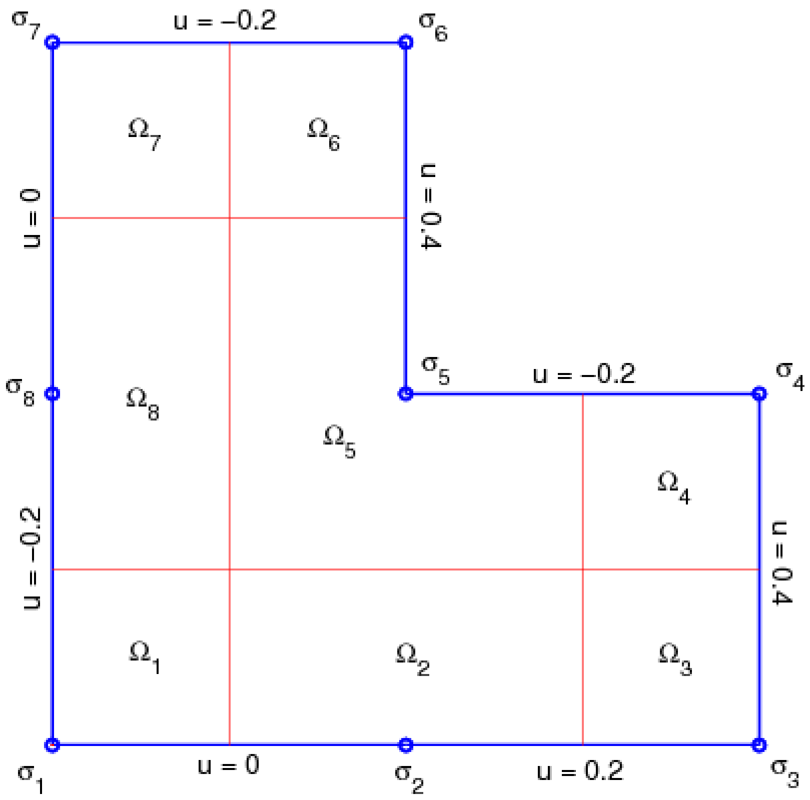

Figure 1-L-shaped domain. Dirichlet boundary conditions.

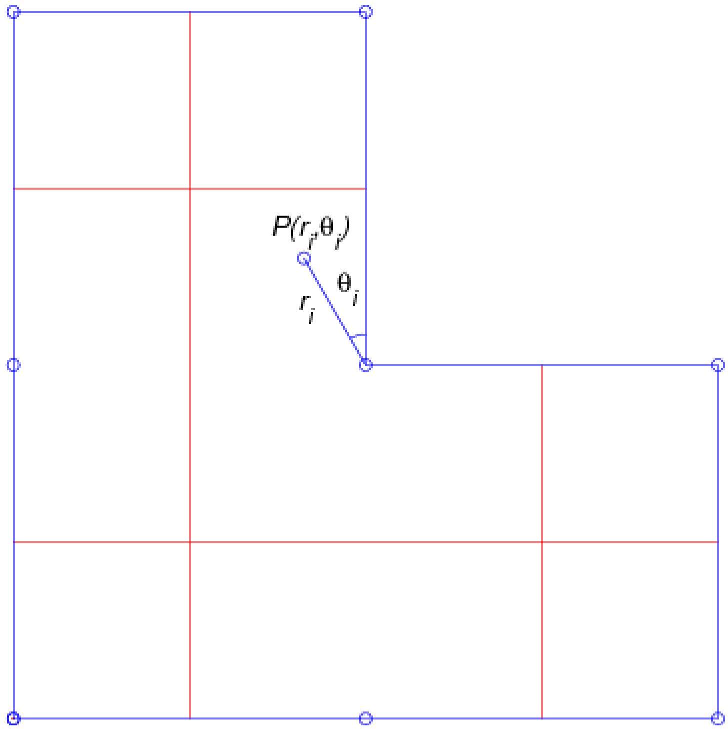

Figure 2- L-shaped domain. Summary of local coordinates.

We solve the Laplace's equation on a polygonal domain using MATLAB software. Boundaries Data can combine conditions of Dirichlet or Neumann.

The method of large singular finite elements comprises three steps [13]:

Step 1: Decomposition of the domain.
Although the domain L-shaped is geometrically symmetrical, it is not possible to replace the problem by an equivalent problem in a reduced domain because conditions on limits are not symmetrical. The polygon must therefore be taken as a whole. The first step of the method consists in dividing the domain into eight sub-domains identified in Figure1, separated by ten sub-borders.

Step 2: Solving auxiliary problems
The number of problems must be equal to the number of sub-domains Ω_i. Therefore, to each sub-domain Ω_i is associated with an origin σ_i, which is a singularity, α_i the opening angle and a local system with polar coordinates (r_i,θ_i) (figure 2).

First Auxiliary Problem

$$\Delta u_1(r_1,\theta_1)=0 \quad (r_1,\theta_1)\in\Omega_1 \tag{1-a}$$

$$u_1(r_1,0=0 \tag{1-b}$$

$$u_1(r_1,\frac{\pi}{2})=-0.2 \tag{1-c}$$

Second Auxiliary Problem

$$\Delta u_2(r_2,\theta_2)=0 \quad (r_2,\theta_2)\in\Omega_2 \tag{2-a}$$

$$u_2(r_2,0=0.2 \tag{2-b}$$

$$u_2(r_2,\pi)=0 \tag{2-c}$$

Third Auxiliary Problem

$$\Delta u_3(r_3,\theta_3)=0 \quad (r_3,\theta_3)\in\Omega_3 \tag{3-a}$$

$$u_3(r_3,0)=0.4 \tag{3-b}$$

$$u_3(r_1,\frac{\pi}{2})=0.2 \tag{3-c}$$

Forth Auxiliary Problem

$$\Delta u_4(r_4,\theta_4)=0 \quad (r_4,\theta_4)\in\Omega_4 \tag{4-a}$$

$$u_4(r_4,0)=-0.2 \tag{4-b}$$

$$u_4(r_4,\frac{\pi}{2})=0.4 \tag{4-c}$$

Fifth Auxiliary Problem

$$\Delta u_5(r_5,\theta_5)=0 \quad (r_5,\theta_5)\in\Omega_5 \tag{5-a}$$

$$u_5(r_5,0)=0.4 \tag{5-b}$$

$$u_5(r_5,\frac{3\pi}{2})=-0.2 \tag{5-c}$$

Sixth Auxiliary Problem

$$\Delta u_6(r_6,\theta_6)=0 \quad (r_6,\theta_6)\in\Omega_6 \tag{6-a}$$

$$u_6(r_6,0)=-0.2 \tag{6-b}$$

$$u_6(r_6,\frac{3\pi}{2})=0.4 \tag{6-c}$$

Seventh Auxiliary Problem

$$\Delta u_7(r_7,\theta_7)=0 \quad (r_7,\theta_7)\in\Omega_7 \tag{7-a}$$

$$u_7(r_7,0)=0 \quad \text{(7-b)}$$

$$u_7(r_7,\frac{\pi}{2})=-0.2 \quad \text{(7-c)}$$

Eighth Auxiliary Problem

$$\Delta u_8(r_8,\theta_8)=0 \quad (r_8,\theta_8)\in\Omega_8 \quad \text{(8-a)}$$

$$u_8(r_8,0=-0.2 \quad \text{(8-b)}$$

$$u_8(r_8,\pi)=0 \quad \text{(8-c)}$$

SOLUTIONS AND DISCUSSION

General solutions of the eight auxiliary problems are written taking into account various boundary conditions specified in figure 1 [12]:

$$u_1(r_1,\theta_1)=-\frac{2\theta_1}{5\pi}+\sum_{i=1}^{\infty}a_{1i}r_1^{2i}\sin 2i\theta_1 \quad \text{(9-a)}$$

$$u_2(r_2,\theta_2)=\frac{1}{5}(1-\frac{\theta_2}{\pi})+\sum_{j=1}^{\infty}a_{2j}r_2^{j}\sin j\theta_2 \quad \text{(10-a)}$$

$$u_3(r_3,\theta_3)=\frac{2}{5}(1-\frac{\theta_3}{\pi})+\sum_{k=1}^{\infty}a_{3k}r_3^{2k}\sin 2k\theta_3 \quad \text{(11-a)}$$

$$u_4(r_4,\theta_4)=-\frac{1}{5}(1-\frac{6\theta_4}{\pi})+\sum_{m=1}^{\infty}a_{4m}r_4^{2m}\sin 2m\theta_4 \quad \text{(12-a)}$$

$$u_5(r_5,\theta_5)=\frac{2}{5}(1-\frac{\theta_5}{\pi})+\sum_{n=1}^{\infty}a_{5n}r_5^{\frac{2m}{3}}\sin\frac{2m}{3}\theta_5 \quad \text{(13-a)}$$

$$u_6(r_6,\theta_6)=-\frac{1}{5}+\frac{6\theta_6}{5\pi}+\sum_{p=1}^{\infty}a_{6p}r_6^{2p}\sin 2p\theta_6 \quad \text{(14-a)}$$

$$u_7(r_7,\theta_7)=-\frac{2\theta_7}{5\pi}+\sum_{q=1}^{\infty}a_{7q}r_7^{2q}\sin 2q\theta_7 \quad \text{(15-a)}$$

$$u_8(r_8,\theta_8)=-\frac{1}{5}(1-\frac{\theta_8}{\pi})+\sum_{j=1}^{\infty}a_{8j}r_8^{j}\sin j\theta_8 \quad \text{(16-a)}$$

Step3: Connecting auxiliary solutions

The third step of the method consists in connecting these auxiliary solutions by imposing the continuity of the function and its normal derivatives all along the various sub-borders Γ_{kl} separating two adjacent sub-domains Ω_k and Ω_l. If conditions on interfaces Γ_{kl} are met, therefore the exact solution to the initial problem will be found. Actually, since we cannot solve an infinite system, we must generally keep to approximate solutions. The approximation result on the one hand, the fact that we must limit the development (9-a) to (16-a) to a finite number of terms, and secondly, that 'we must content ourselves, with rare exceptions, a connection imperfect. Approximate solutions have been obtained by limiting the series used in general solutions. The number of terms kept in each of the sums is chosen according to the principle proposed by Descloux and Tolley [15] aiming at representing the approximate solutions using functions of degrees as uniform as possible. This is achieved by keeping more coefficients for sub-domains with larger openings.

Approximate auxiliary solutions are all of 2N degrees (N being the number of coefficients retained for the square sub-domains which opening angle is $\pi/2$). Finally, the total number of unknown parameters a_{kl}, which value can be freely chosen will be: $(5\times1+2\times2+3\times1)N=12N$

This allows getting the following approximate solutions:

$$u_1(r_1,\theta_1) = -\frac{2\theta_1}{5\pi} + \sum_{i=1}^{N} a_{1i} r_1^{2i} \sin 2i\theta_1 \quad (9\text{-b})$$

$$u_2(r_2,\theta_2) = \frac{1}{5}(1-\frac{\theta_2}{\pi}) + \sum_{j=1}^{2N} a_{2j} r_2^{j} \sin j\theta_2 \quad (10\text{-b})$$

$$u_3(r_3,\theta_3) = \frac{2}{5}(1-\frac{\theta_3}{\pi}) + \sum_{k=1}^{N} a_{3k} r_3^{2k} \sin 2k\theta_3 \quad (11\text{-b})$$

$$u_4(r_4,\theta_4) = -\frac{1}{5}(1-\frac{6\theta_4}{\pi}) + \sum_{m=1}^{N} a_{4m} r_4^{2m} \sin 2m\theta_4 \quad (12\text{-b})$$

$$u_5(r_5,\theta_5) = \frac{2}{5}(1-\frac{\theta_5}{\pi}) + \sum_{n=1}^{3N} a_{5n} r_5^{\frac{2m}{3}} \sin \frac{2m}{3}\theta_5 \quad (13\text{-b})$$

$$u_6(r_6,\theta_6) = -\frac{1}{5} + \frac{6\theta_6}{5\pi} + \sum_{p=1}^{N} a_{6p} r_6^{2p} \sin 2p\theta_6 \quad (14\text{-b})$$

$$u_7(r_7,\theta_7) = -\frac{2\theta_7}{5\pi} + \sum_{q=1}^{N} a_{7q} r_7^{2q} \sin 2q\theta_7 \quad (15\text{-b})$$

$$u_8(r_8,\theta_8) = -\frac{1}{5}(1-\frac{\theta_8}{\pi}) + \sum_{j=1}^{2N} a_{8j} r_8^{j} \sin j\theta_8 \quad (16\text{-b})$$

We are therefore connecting the solutions of auxiliary problems according to least squares method, i.e. we calculate coefficients a_{kl} that allow minimizing the function:

$$I(a_{mn}) = \sum_{i<j} \int_{\Gamma_{ij}} \left[(u_i(a_{ik}) - u_j(a_{jl}))^2 + \left(\frac{\partial u_i(a_{ik})}{\partial n_i} + \frac{\partial u_j(a_{jl})}{\partial n_j} \right)^2 \right] ds_{ij} \quad (17)$$

These coefficients are solution of the linear algebraic system positive definite square matrix of order 12N with 12N unknowns classically called normal equations of Gauss:

$$\frac{\partial I(a_{mn})}{\partial a_{kl}} = O \quad m,\, k = 1,2,\ldots,\, 8; \qquad n,l = 1,\ldots.12N \quad (18)$$

The accuracy of approximate solutions is directly linked to the quality of the connection of solutions to auxiliary problems. It is therefore natural to characterize its precision by measuring the imperfections of continuity conditions. We will use the overall connecting error definite by (19):

$$\eta = \sum_{k<l} \frac{1}{S_{kl}} \int_{\Gamma_{kl}} \left[(u_k - u_l)^2 + \left(\frac{\partial u_k}{\partial \upsilon_k} + \frac{\partial u_l}{\partial \upsilon_l} \right)^2 \right] ds_{kl} \quad (19)$$

Where ds_{kl} is the element which arch length is Γ_{kl}, S_{kl} its length and υ_k and υ_l the normals to the sub-border separating both adjacent sub-areas. If the overall error is null, the approximate solution got aligns with the exact solution.

Mode of convergence of the method of large singular finite elements is exponential. Indeed, the base 10 logarithm of the overall error decreases linearly with 12N as shown in Figure 3. Moreover, this curve of the overall error allows us to say, for example, that by keeping 288 coefficients a_{kl}, the numerical results are obtained with an absolute accuracy of less than 10^{-11}.

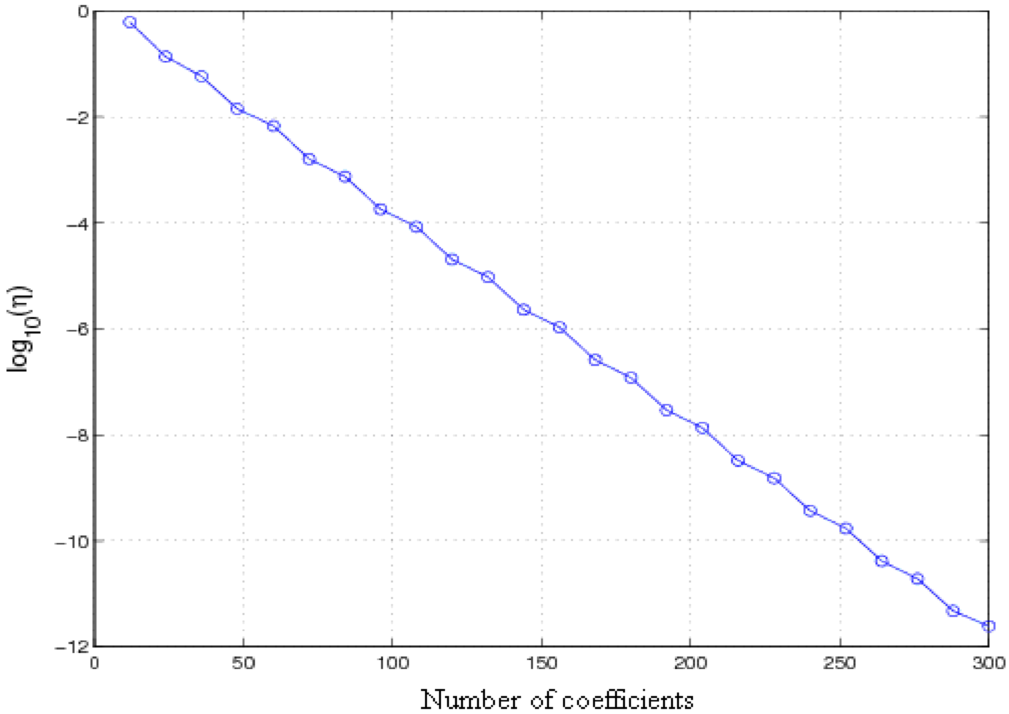

Figure 3- Evolution of the overall error according to the number of coefficients a_{kl} retained.

In order to compare results obtained using the method of large singular finite elements with those provided by the method of conventional finite elements in its standard form, the problem was solved using COMSOL software.

The comparison of both methods is summarized in few graphics shown in figures 4, 5 and 6. In figure 4, approximations of the function u and its derivatives were presented all along circles centered on singularity σ_5 (figure 1). The full lines have been obtained using the method of large singular finite elements by keeping 288 coefficients a_{kl} and the small circles are the results from the finite elements method for a range of 367,617 degrees of freedom. Figure 5 is similar to the previous one, but only data on function u for the four circles are there presented.

Finally, the bi-logarithmic chart in figure 6 gives the maximal gaps between results obtained using the method of large singular finite elements and the finite elements method all along various circles centered in σ_5 according to their radius.

Results obtained by both methods are in line with the whole domain of calculation. However, if we examine the numerical values obtained near the singular points (discontinuity points of the function), it appears that the method of large singular finite elements is significantly well accurate enough. If gaps between both methods are around 10^{-4} over a circle which radius is 5.10^{-1}, it increases to 10^{2} for a circle which radius is 10^{-3}, then to 10^{4} if the radius of the circle is 10^{-5}.

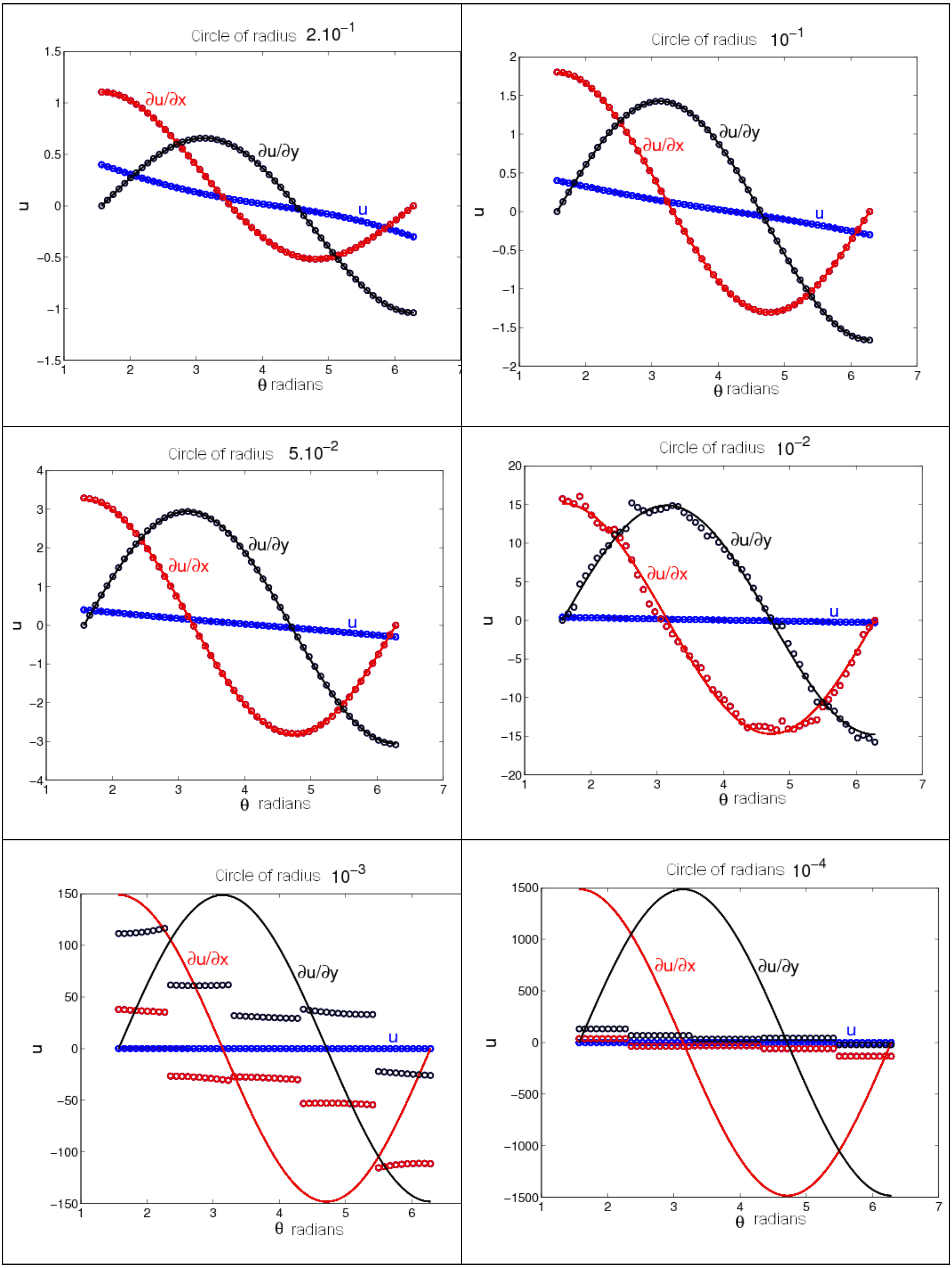

Figure 4 –Comparing the values of u (blue), $\frac{\partial u}{\partial x}$ (red) and $\frac{\partial u}{\partial y}$ (black) obtained through the method of large singular finite elements (full lines) and finite elements method (circles).

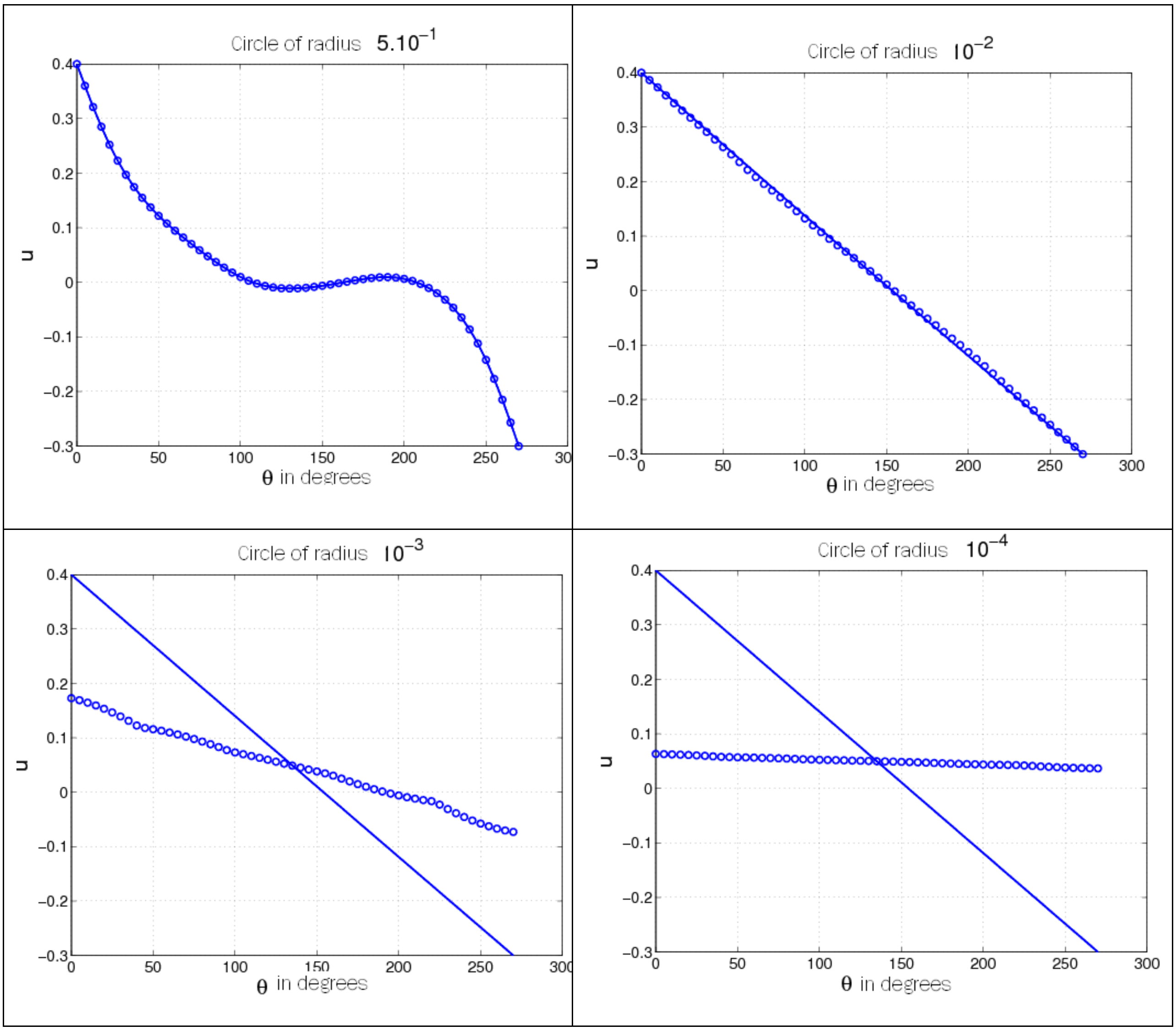

Figure 5- Comparing the values of *u* obtained using the method of large singular finite elements (full lines) and finite elements method (circles).

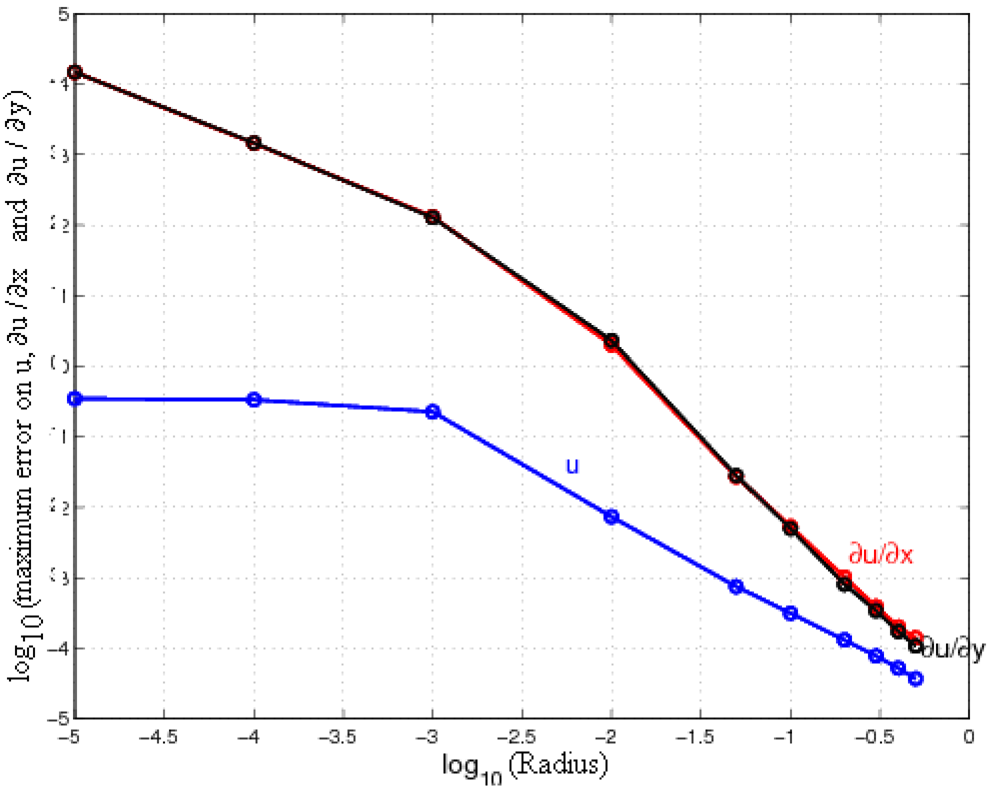

Figure 6- Maximal local gap between the values of *u* and its derivatives calculated using the method of large singular finite elements and finite elements method all along the various circles centered on the singularity σ_5

CONCLUSION

Results obtained by the method of large singular finite elements and finite elements method are in line all over the domain. However, by checking the numerical values obtained near the singular points, this shows that the method of large singular finite elements is well accurate enough. The method of large singular finite elements takes the existence of singularities into account, while trying to find analytical solutions near them; this allows therefore getting without further formulation, derivative values. Moreover, the mode of convergence of this method is exponential.

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REFERENCES

[1] 0. Maurice, A. Reineix, P. Hoffmann, B. Bernard and P. Pouliguen, *Adv. in Appl. Sci. Res*. **2011**, 2 (5), 439.
[2] D. S. Chauhan and V. Kumar, *Adv. in Appl. Sci. Res*. **2012**, 3 (1), 75.
[3] R. Saravana, S. Sreekanth, S. Sreenadh and S. Hemadri Reddy. *Adv. in Appl. Sci. Res*. **2011**, 2 (1), 221.
[4] V. Sri Hari Babu and G.V. Ramana Reddy, *Adv. in Appl. Sci. Res*. **2011**, 2 (4), 138.
[5] R.E Barnhill and J. R. Whiteman, Singularities due to re-entrant boundaries in elliptic problem. in L. Collatz (ed.), I.S.M.N. 19, Birkhauser-Verlag, Basel, **1974.**
[6] A. F. Emery, *Trans. ASME* (C), **1973**, 95, 344.
[7] G. J. Fix, *Math. Mech*., **1969**, 18, 645.
[8] Z.C. Li and T.T. Lu, *Math. Comput. Model*., **2000**, 31, 97.
[9] H. Motz, *Quart. Appl. Math*., **1946**, 4, 371.
[10] G. Strang and G.Fix, Analysis of the finite element method. Prentice Hall, **1973**.
[11] R. Wait and A. R. Mitchell, *J. Comput. Phys*. **1971,** 8, 45.
[12] Z. C. Li, T. T. Lu, H. Y. Hu and A. H. D. Cheng. *Eng. Anal. Bound. Elem.*, **2005,** 29, 59,

[13] L. Fox, P. Henrici and C. B. Moler, *SIAM*, *J. Numer. Anal*. **1967**, 4, 99.
[14] M. D. Tolley, PhD Thesis, Université Libre de Bruxelles, (Bruxelles, Belgium), **1977**.
[15] J. Descloux and M. D. Tolley, *Comput. Method. Appl. Mech. Eng*, **1983**, 39 (1), 37.